Factor Copulas: Totally External Defaults

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Abstract

In this paper we address a fundamental problem of the standard one factor Gaussian Copula model. Within this standard framework a default event will have a large impact on the default probability of the survivors, through a shock in available information on the common factor. Moreover this effect is larger for defaults occurring instantaneously. In this paper it is shown that this problem is caused by linearly combining the common factor and idiosyncratic terms.

In this paper we propose an extended model, which overcomes this problem. An extra idiosyncratic term is introduced which models totally external default risk. Here one should think of default events due to fraud, or legal issues. The most noticeable default events over the last couple of years have been Enron, Worldcom and Parmalat. All of these defaulted due to totally external causes.

The occurrence of such default events do not increase the available information on the common factor driving correlated defaults.

In addition it is shown that this extended model provides a plausible explanation for the observed compound correlation smile, or equivalently, the base correlation skew. Moreover, this model requires two additional parameters with which one can control the shape of the so-called base correlation skew.

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1 Introduction

Over the last decade, the growth in the credit derivatives market has been enormous. The market in credit default swaps on single names has become the very liquid, with entire termstructures of CDS quotes available for a large amount of reference entities. These credit default swaps are now treated as the plain vanilla instruments, the prices of which are used to calibrate models for valuing more complex credit derivatives.

Apart from the credit derivatives contingent on the default behavior of a single reference entity, the market for basket related credit derivatives, or multi-name credit derivatives, has shown tremendous growth over the last couple of years. Especially the market in tranche default swaps has grown explosively, where quotes for standard tranches on standard reference baskets become more widely available. In practice the one factor Gaussian factor copula has become extremely popular. However, it has become clear that this standard model is not capable of explaining the observed prices of different tranches on a single basket. Due to the increased liquidity in some standard CDO products, it is crucial that any model used for pricing such products should price the standard tranches equal to their market values. In order to achieve this an ad-hoc method has become popular, known as the Base Correlation approach, due to McGinty, Beinstein, Ahluwalia, and Watts (2004). This method can not be regarded as a model explaining the observed market quotes, but rather as a fix of the standard one factor Gaussian copula model. It has also provided a tool for visualizing the correlation structure embedded in the market quotes. In order to explain the market implied correlation structure some extensions to the standard one factor Gaussian model have been proposed. For instance, Andersen and Sidenius (2004) have extended the model by allowing for random recovery rates and random factor loadings.

In this paper we take a slightly different approach. Apart from the fact that it can not explain observed market quotes, one of the main fundamental problems of the one Factor Gaussian copula model is the fact that a default event of a certain name has a lot of influence on the default probabilities of correlated surviving names. However the default event of Parmalat, for instance, did not result in a large jump in CDS rates of other names. This fundamental problem in the model becomes even more apparent for a default event close to the time of modelling. This problem of unrealistically high default correlation effects for instantaneous defaults has been documented in Rogge and Schönbucher (2003).

We show that this behavior is inherent to models which linearly combine common factor risk and idiosyncratic risk sources. By proposing a simple and elegant extension to the standard model we achieve two things: first the undesirable correlation effect of short term defaults is not present in the extended model. Second, the model is capable to explain the observed market quotes for standard tranches.

The outline of this paper is as follows. In the following section we start with a review of the standard one factor Gaussian copula model and discuss two of its main weaknesses. First the unrealistic correlation effect for instantaneous defaults is addressed. Next we touch upon the fact that the model is incapable of explaining observed market quotes. Here the Base Correlation method due to McGinty, Beinstein, Ahluwalia, and Watts (2004) is also discussed. In section 4 the model of this paper is discussed. It is shown how this model overcomes the problem of instantaneous defaults and we argue how this model might explain the Base Correlation skew observed in the market. In section 5 some results are shown. First a comparison is made of the standard one factor Gaussian copula model and the extended model. Apart from this the Base Correlation skew implied by the extended model is considered for different parameter settings. Finally, section 7 concludes.

2 Factor Copula

In this section a brief review of the factor copula model is given. In addition two mayor drawbacks of the approach are discussed.

The one factor Gaussian copula is a special case of the more general Gaussian copula, due to Li (2000). The default event of a single name is modelled by means of a latent standard normal variable. Correlation is imposed by correlating these latent variables and under the one factor copula this is done by means of a common factor.

$$X_i = \rho_i Y + \sqrt{1-\rho_i^2}\xi_i$$

In this formula, X_i denotes the latent variable for name i. Y denotes the common factor and ξ_i denotes an idiosyncratic term. The terms ξ_i, $i = 1, \ldots, M$ and Y are all *i.i.d.* standard normal variables. The parameter ρ_i determines the correlation of the latent variable with the common factor. Thus the correlation between latent variables of two different names is given by $\rho_i \rho_j$.

It is well-known that any random variable from a continuous distribution can be transformed to a uniform variable by using the distribution function. This can be used to relate default events to the latent variable such that the marginal default distribution, $p_i(t)$, is maintained:

$$\tau_i \leq t \iff X_i \leq \Phi^{-1}\left(p_i\left(t\right)\right) \tag{1}$$

This relation can be used to translate calculations involving default times into calculations involving standard normal variables, which are correlated by means of a single factor. Now the power of the one factor copula model lies in the fact that one can condition on the common factor, after which all remaining sources of risk become independent. Thus the default probability for name i given the realized value for the common factor Y is given by:

$$\begin{aligned} \Pr\left(\tau_i \leq t \middle| Y = y\right) &= \Pr\left(X_i \leq \Phi^{-1}\left(p_i\left(t\right)\right) \middle| Y = y\right) \\ &= \Pr\left(\rho_i y + \sqrt{1-\rho_i^2}\xi_i \leq \Phi^{-1}\left(p_i\left(t\right)\right)\right) \\ &= \Pr\left(\xi_i \leq \frac{\Phi^{-1}\left(p_i\left(t\right)\right) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}\left(p_i\left(t\right)\right) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) \end{aligned} \tag{2}$$

Equipped with these independent conditional default probabilities one can determine the conditional loss distribution of the entire basket. This can be achieved by applying a grid method, as described in Andersen, Sidenius, and Basu (2003). This is a recursive algorithm, where at each step a new name from the basket is brought in. Every name can either default or survive during the period under consideration, resulting in an loss increase, or the loss remaining unchanged. In the special case when the recovery rates are equal, one can restrict to merely modelling the number of defaults. In this case the grid method of Andersen, Sidenius, and Basu (2003) reduces to a binomial tree with number of steps equal to the number of names. Every up-move corresponds to a default event and every down-move to survival. The values in the endnotes than give the probabilities of exactly k default events, for k running from 0 to the total number of names in the basket.

After the desired distribution has been determined, one can integrate out over the common factor to obtain the unconditional loss distribution of the entire basket. Equipped with this loss distribution one can easily determine prices for tranche default swaps, which provide protection against losses on CDO tranches. More information on pricing can be found in Laurent and Gregory (2003), or Hull and White (2004).

2.1 Curse of Instantaneous Defaults

One mayor drawback of the one Factor Gaussian copula framework is the limited complexity of the correlation structure. This is reflected in the way default events affect the default probability of surviving names. The paper by Schönbucher and Schubert (2001) gives a detailed analysis of the effects of defaults and survival on the default behavior of surviving names.

This correlation effect is especially relevant for default events in the near future. Consider a fast default event for a certain name, which not has an extremely high CDS spread at time zero. This fast default event means that the corresponding latent variable must have been extremely low. Due to the fact that the standard normal density tends to zero very fast for values far from 0, one can expect that the low value of the latent variable was caused by low values for both the common factor as well as the idiosyncratic factor. Thus conditional on a fast default event, it is very likely that the realization of the common factor is very small. This has dramatic consequences for the remaining names, as they will be much more likely to default in the immediate future.

In addition, at time zero one should be able to judge the current state of the economy to a large extend. However, as was just shown, a fast default event directly results in the state of the economy being negative. This fundamental shortcoming of the model is caused by the fact that, the model is not able to explain defaults, which are not caused by macro economic behavior but are entirely external. I.e. every unexpected fast default event must be caused by a bad state of the common factor and thus increases the default probability of survivors. This problem is most visible for fast defaults, but any default event will directly give a shock in available information on the common factor.

2.2 Base Correlations

A second mayor drawback of the standard one factor Gaussian approach is more of a practical nature. In recent years, quotes on CDOs become more readily available, allowing one to calibrate model parameters to these observed market quotes. When adopting this approach for various tranches on a certain basket one can calculate a compound correlation for each tranche. This is defined as that correlation which, when used in the model, gives a fair premium equal to the observed market quote. Note that this requires the additional assumption that $\rho_i \equiv \rho, \forall i$, in (2).

Under the one factor Gaussian copula model one would expect the compound correlation being equal for every single tranche on a certain basket. However, this is not the case as the compound correlations, when plotted against the detachment level, show a figure which resembles a smile. In the following figure such a compound correlation smile is presented, based on market quotes for tranches on the DJ CDX basket on October the $18^t h$, 2004.

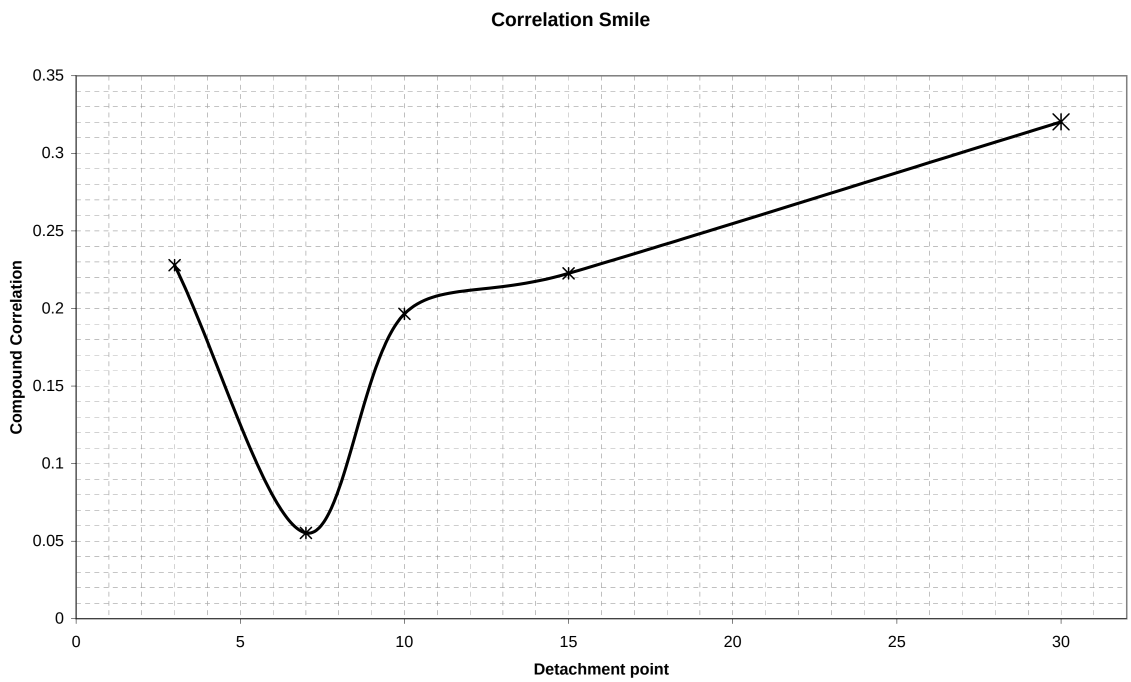

Figure 1: Compound Correlations plotted against the detachment level of the tranches. Quotes on standard tranches on the DJ CDX basket are used from the 18^{th} *of October 2004.*

From this figure one can see the so-called correlation smile. The compound correlation for the equity tranche is around 23% after which the compound correlation decreases to about 5% for the tranche with attachment and detachment levels at 3% and 7%, respectively. Next, one can observe upward sloping behavior for the more senior tranches.

In contrast to implied volatilities in the Equity derivatives market, one can not just use interpolation methods to obtain correlations required for pricing other tranches. The problem here is that these compound correlations are a function of both attachment point as well as detachment point. Fortunately a clever observation, used in McGinty, Beinstein, Ahluwalia, and Watts (2004), allows one to write the expected loss on a certain tranche as the difference between the expected loss of two tranches with zero attachment. In this way one can determine the expected loss for each of the available detachment levels. These expected losses can again be translated to a corresponding correlation parameter, when using the standard factor Gaussian copula model. These latter correlations are known as the base correlations and these are usually upward sloping in detachment level. As an example we have plotted the base correlations against the detachment level using market data of October 18^{th}, 2004. This results in the following figure:

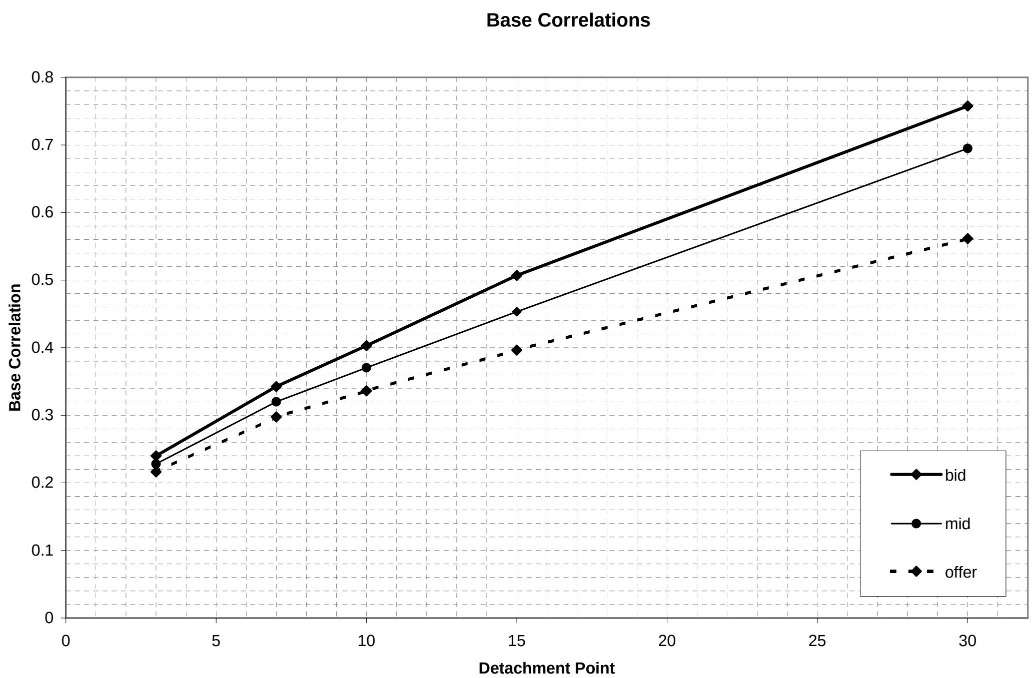

Figure 2: Base Correlations plotted against the detachment level of the tranches. Quotes on standard tranches on the DJ CDX basket are used from the 18^{th} *of October 2004.*

In the figure one can observe the close to linear relationship between base correlations and attachment level. The three different lines are for bid, offer and mid quotes.

It should be stressed that this approach can not be seen as a proper model, but merely as a method to visualize market quotes in terms of correlations.

3 Factor Copula Extensions

In order to explain the observed market behavior different extensions to the standard factor copula model have been proposed. In this section some of these extensions are discussed.

For instance, Andersen, Sidenius, and Basu (2003) use a student t distribution to model the latent variables. Hull and White (2004) model both the common factor as well as the idiosyncratic terms by means of student t distributions. Here one should note that both approaches are different as a linear combination of student t distributed variables does not result in random variable with a student t distribution.

Other research, such as Andersen and Sidenius (2004) extend the model by allowing for random recovery.

Another extension proposed in the paper by Andersen and Sidenius (2004) is to allow the factor weights ρ to depend on the state of the economy, or the common factor, Y. In this case one can model high correlations when the economy is in a bad state and low correlations when the common factor is high.

However, all these proposed extensions do not solve the fundamental problem of unrealistic correlation effects in case of default events in the immediate future. All proposed extensions focuss on a linear combination of a common factor and idiosyncratic terms. Moreover, the random factor weights approach of Andersen and Sidenius (2004) is usually applied with factor weights decreasing in the value of the common factor. Thus for instantaneous defaults, the common factor realization is likely to be low and thus correlation is high. This increases the problem of instantaneous defaults even further.

4 Totally External Defaults

As discussed in 2.1 default events in the near future have large consequences on the surviving names. It was shown that these fast unexpected default events must have been caused by jointly low realizations of the common factor and the idiosyncratic term. The problem lies in the fact that the standard one factor Gaussian copula function linearly combines both these random factors in the construction of the latent variable. Based on this observation we propose an extension, where two latent variables are used for each name. One modelling the standard micro economic behavior of the firm, while the other can be regarded as modelling external behavior.

$$\begin{aligned} X_i &= \rho_i Y + \sqrt{1-\rho_i^2}\xi_{i,1} \\ Z_i &= \mu_i + \sigma_i \xi_{i,2} \end{aligned} \tag{3}$$

$$\tau_i \leq t \iff \min\{X_i, Z_i\} \leq \chi_i(t)$$

Here Y, $\xi_{i,1}$ and $\xi_{i,2}$ all have independent standard normal distributions. Default takes place in case the minimum of the two latent variables is smaller than some threshold, $\chi_i(t)$. Similarly as for the standard one factor Gaussian copula, one can calibrate the threshold such that the marginal distribution is matched.

One can interpret the term X_i as a factor modelling market risk. Here correlation is modelled by means of the common factor Y. Apart from this the term Z_i can be interpreted as modelling external default risk, not caused by standard economical behavior of the firm, such as fraud. Thus, by using the minimum of two factors, a default event caused by external factors does not lead to the increase in information on the common factor. This is exactly what we want to achieve.

The parameters μ_i and σ_i model the likelihood of a default event being caused by external issues. Large values for σ_i correspond to a large likelihood of defaults being caused by some of these events. Thus companies which are not trusted can be given a large value for σ. For μ it is the other way around. Large values for μ decrease the likelihood of defaults due to fraud. With this model, one can thus avoid the well-known problem of the standard factor copula, where short term defaults have extreme effects on the surviving names. A default occurs in case either of the two factors is smaller than the threshold. For instantaneous defaults this threshold is extremely low. Thus in case of a fast default it follows that the minimum of the two factors in (3) is extremely small. Thus in case σ_i is larger than 1, i.e. the variance of Z_i is larger than that of X_i, the default is more likely to have an external cause and hence provides no information on the value of the common factor and thus will not influence overall default behavior. This effect is a highly desirable characteristic of the model. At initialization, one has an indication of the current state of the economy and thus an instantaneous default should not be caused by the common factor. When comparing this with the standard factor copula, it can be argued that a fast default is directly related to a very small value of the latent variable. Due to the fact that the latent variable is a linear combination of the common factor and the idiosyncratic term, one can conclude that both of these random terms should be very small. This results from the rapid decrease to zero in the tails of the normal distribution.

In further analysis we will be interested in default probabilities, either conditional on the realization of the common factor or unconditionally. In order to simplify notion, we first introduce the term $\kappa_i(t)$ which gives the probability of the external default risk factor being smaller than the threshold at time t for name i:

$$\kappa_i(t) \equiv Pr\left(Z_i < \chi_i(t)\right) = \Phi\left(\frac{\chi_i(t) - \mu_i}{\sigma_i}\right) \tag{4}$$

Since $\chi_i(t)$ denotes the threshold for name i corresponding to default before time t, One can easily determine the default probability:

$$\begin{aligned} Pr(\tau_i \leq t) &\equiv Pr\left(\min\left\{X_i, Z_i\right\} \leq \chi_i(t)\right) \\ &= 1 - Pr\left(X_i > \chi_i(t), Z_i > \chi_i(t)\right) \\ &= 1 - Pr\left(X_i > \chi_i(t)\right) Pr\left(Z_i > \chi_i(t)\right) \\ &= 1 - \left(1 - \Phi\left(\chi_i(t)\right)\right) \cdot \left(1 - \kappa_i(t)\right) \\ &= \left(1 - \kappa_i(t)\right) \cdot \Phi\left(\chi_i(t)\right) + \kappa_i(t) \end{aligned} \tag{5}$$

Here we have used the independence between the market risk variable X_i and the external risk variable Z_i. From this equation one can easily determine the threshold levels $\chi_i(t)$, such that the default probability of name i matches the desired level. Note that, in general, this requires a numerical root finding routine.

Once the thresholds are determined, it is straightforward to determine the conditional default probabilities, i.e. the probability of default conditional on the realization of the common factor.

$$Pr(\tau_i \leq t\,|Y = y) = \left(1 - \kappa_i(t)\right) \cdot \Phi\left(\frac{\chi_i(t) - \rho_i y}{\sqrt{1 - \rho_i^2}}\right) + \kappa_i(t) \tag{6}$$

Pricing multi-name credit derivatives under this model now goes along the same lines as for the standard one factor Gaussian Copula model, (2). I.e. one can condition on the realization of the common factor, determine the default probabilities as given in (5) and apply the grid method of Andersen, Sidenius, and Basu (2003) to obtain the conditional loss distribution. After integrating out over the distribution of the common factor, one can then directly price credit derivatives, such as tranche default swaps.

Note that the model in (3) can easily be extended to allow for random recovery or random factor loadings, as done in Andersen and Sidenius (2004). In this way one can model large default correlations and low recovery rates when the economy is in a recession and low correlations and high recovery rates when the economy is in an expansion.

Also, one can investigate the model using different distributions for the external default risk factor, Z. For instance a t-distribution might be considered. In this paper we merely focuss on the model as stated in (3) as the parameters μ and σ allow for a large amount of additional flexibility with respect to the standard model. Moreover, in order to avoid too much complexity, throughout the remainder of this paper we assume that the parameters ρ_i, μ_i and σ_i are equal for all names.

Under the proposed model, a single default event does not directly give much information on the state of the economy. The default event might have an external cause. In case of a larger number of default events, it becomes more and more likely that the common factor has a low realization or equivalently, the economy is in a bad state. Thus on the one hand, more independence is created at the short end of the total loss, while on the other hand default correlation is larger at the far end of the portfolio loss. This is exactly the kind of behavior which is reflected in the market for CDOs, due to the upward sloping nature of the Base Correlation skew.

Furthermore, when considering the extended model as stated in (3), one can observe that actual default correlation will always be less than that of the corresponding standard factor copula model. The inclusion of the external default risk factor will bring correlation down. Thus the Base Correlation for any detachment level will always be smaller than ρ^2. This important observation can be used when one wants to estimate model parameters from observed market quotes. One can apply the Base correlation method to the observed quotes and the correlation with the common factor should be at least as large as the largest available base correlation, usually the one corresponding to the largest detachment level.

In order to successfully understand the model, it is important to gain insight in the probability that a default event is caused by either the market risk factor or the external default risk factor. We are thus interested in:

$$Pr\left(Z \leq X \left| \min\{X, Z\} \leq \chi\right.\right) \quad = \quad \frac{Pr\left(Z \leq X, \min\{X, Z\} \leq \chi\right)}{Pr\left(\min\{X, Z\} \leq \chi\right)} \tag{7}$$

Note that for brevity both the dependence on the particular name and time has been removed. The denominator of the expression above is the default probability for threshold χ and has been presented in (5). Evaluating the numerator is what remains. Again the independence of the factors is used, which allows the multivariate distribution to be written as the product of the marginals.

$$
\begin{aligned}
& Pr\left(Z \leq X, \min\{X, Z\} \leq \chi\right) && (8) \\
= & \int_{-\infty}^{\infty} \phi(x) \int_{-\infty}^{\min\{x,\chi\}} \phi_{\mu,\sigma}(z) dz dx \\
= & \int_{-\infty}^{\chi} \phi(x) \int_{-\infty}^{x} \phi_{\mu,\sigma}(z) dz dx + \int_{\chi}^{\infty} \phi(x) \int_{-\infty}^{\chi} \phi_{\mu,\sigma}(z) dz dx \\
= & \int_{-\infty}^{\chi} \phi(x) \cdot \Phi\left(\frac{x-\mu}{\sigma}\right) dx + \int_{\chi}^{\infty} \phi(x) \cdot \Phi\left(\frac{\chi-\mu}{\sigma}\right) dx \\
= & \Phi_2\left(\frac{-\frac{\mu}{\sigma}}{\sqrt{1+\frac{1}{\sigma^2}}}, \chi; \frac{-\frac{1}{\sigma}}{\sqrt{1+\frac{1}{\sigma^2}}}\right) + \Phi\left(\frac{\chi-\mu}{\sigma}\right) \cdot (1-\Phi(\chi))
\end{aligned}
$$

Here Φ_2 denotes the bivariate cumulative normal distribution function. To obtain the result in the final line, one can make use of the results in the appendix of Andersen and Sidenius (2004)

4.1 Large Portfolio Limit

An approximation which can be used in case the number of names in the portfolio is large is the large homogenous pool approximation due to Vasicek (1991). In case one considers a pool with an infinite number of equivalent names, there is no uncertainty left once the realization of the common factor is known, due to the law of large numbers. Thus, when knowing the realization of the common factor, one knows exactly the fraction of names of the portfolio which have defaulted. In order to determine this for the model with external default events we follow along the same lines as the derivation in Schönbucher (2003). Let L denote the fraction of names of the portfolio having defaulted.

$$\begin{aligned} Pr(L \leq l) &= \int_{-\infty}^{\infty} Pr(L \leq l\,|Y = y)\phi(y)dy \qquad (9) \\ &= \int_{-\infty}^{\infty} I\left(\kappa(t) + (1-\kappa(t))\cdot\Phi\left(\frac{\chi(t)-\rho\cdot y}{\sqrt{1-\rho^2}}\right) \leq l\right)\phi(y)dy \\ &= \int_{y^*}^{\infty} 1\cdot\phi(y)dy = \Phi(-y^*) \end{aligned}$$

Here $I(\cdot)$ denotes the indicator function. What remains is to determine the level y^* at which the conditional default probability exactly matches the loss fraction l. This is just a matter of inverting. The resulting distribution is thus given as:

$$Pr(L \leq l) = \Phi\left(\frac{\sqrt{1-\rho^2}\cdot\Phi^{-1}\left(\frac{l-\kappa(t)}{1-\kappa(t)}\right) - \Phi^{-1}\left(\frac{p(t)-\kappa(t)}{1-\kappa(t)}\right)}{\rho}\right) \qquad (10)$$

Here one can note the close resemblance with the large homogenous pool result for the standard one factor Gaussian copula model. Both l and $p(t)$ have been transformed using the probability of an external default event. Note that this formula should be applied for $l \geq \kappa(t)$. This is caused by the fact that no matter what the realization of the common factor will be, the number of defaults is bounded from below by $\kappa(t)$ due to external default risk.

4.2 Extending the Model

The model described so far has been presented in its simplest form and can easily be extended in various ways. Therefore, consider both the market risk variable X as well as the external default risk variable Z to have general distributions. As in earlier derivations, it is useful to define $\kappa_i(t) \equiv Pr\left(Z_i \leq \chi_i(t)\right)$. I.e. $\kappa_i(t)$ determines the probability of an external default event for name i, before time t. The unconditional probability of default as well as the conditional probability of default can then be written as follows:

$$Pr(\tau_i \leq t) = (1-\kappa_i(t))\cdot Pr\left(X_i \leq \chi_i(t)\right) + \kappa_i(t) \qquad (11)$$

$$Pr(\tau_i \leq t\,|Y = y) = (1-\kappa_i(t))\cdot Pr\left(X_i \leq \chi_i(t)|\,Y = y\right) + \kappa_i(t)$$

Thus the model can be extended either by means of the distribution for the market risk variable, or by the distribution of the external risk variable.

When allowing for more general distributions for the market risk variable, different alternatives which have been discussed in section 3 can be used directly. For instance one can use a student t distribution for the market risk variables as was discusses in Andersen, Sidenius, and Basu (2003). A related alternative, due to Hull and White (2004) is to construct the market risk factors by linearly combining a common factor and an idiosyncratic term, both having a student t distribution. Also, one can easily extend the model to allow for stochastic correlations, or random factor loadings. The paper by Andersen and Sidenius (2004) gives a detailed derivation of the required formulae.

Apart from these extensions, one can focuss on the distribution of the external risk factor. The normal distribution used throughout this paper might not adequately model the likelihood of default events being due to external causes. Moreover the behavior over time should be carefully chosen. For instance, one might want to get rid of the problem of instantaneous defaults, but at the same time allow for a long term equilibrium with a fixed relationship between defaults due to market risk and default due to external causes.

5 Numerical Results

In this section the standard one factor Gaussian copula model is compared with our extended model. The first part of the section focusses on the curse of instantaneous defaults. Some simple tests are performed to compare the implications of a fast default event of a single name for the names which have survived.

The second part of this section focusses on the differences between the standard model and the proposed extended model for tranche pricing. This is done by determining the Base Correlation skew generated by the extended model.

5.1 Instantaneous Defaults

As discussed in subsection 2.1, the standard one factor Gaussian copula model yields undesirable correlation effects, especially for fast default events. In order to investigate this, we consider a simple basket of two names, both having recovery rate of 40% and a CDS premium of $50BP$. The marginal default distributions are modelled by means of a constant default intensity for both names, approximately $83BP$. [1] Under these assumptions we will investigate the effect of a default of one of the names on the default probability of the other name. More specifically, we condition on default of the second name and survival of the first name, up to time t and investigate the default probability of the first name within the next year, thus up to $T \equiv t + 1Y$. This probability can be calculated:

$$Pr\left(\tau_1 \leq T \,|\tau_1 > t, \tau_2 \leq t\,\right) \quad = \quad \frac{Pr\left(t < \tau_1 \leq T, \tau_2 \leq t\right)}{Pr\left(\tau_1 > t, \tau_2 \leq t\right)} \tag{12}$$

Both numerator and denominator of the r.h.s. of (12) can easily be determined by first conditioning on the realization of the common factor. The default probabilities of both names will then be independent, allowing for straightforward derivations. The conditional probabilities can then be integrated out and divided to obtain the desired probability. This integration can either be done numerically or in closed form. The closed form formula can be derived by applying the results in the appendix of Andersen and Sidenius (2004). We will not go into detail here as the exercise is merely performed in order to investigate the default correlation effects of both the standard model and the extended one.

We investigate this probability for different times of default, t. This allows us to see the implications of the extended model on the correlation effect of fast default events. The following figure displays three different curves. One gives the results of the standard factor Gaussian copula, while the other curves result from the extended model. In one case we have given the external default risk factor, Z a standard normal distribution. In the other case its distribution is normal with mean 20 and standard deviation 10. In all cases the copula correlation $\rho_1 \cdot \rho_2$ has been set to 20%. Further one should keep in mind that the 1Y default probability of a single name in case of independence is about $83BP$.

[1] *The relationship between a CDS spread* π *, the recovery rate R and default intensity* λ *, is approximately given by* $\lambda \approx \pi \cdot (1 - R)$

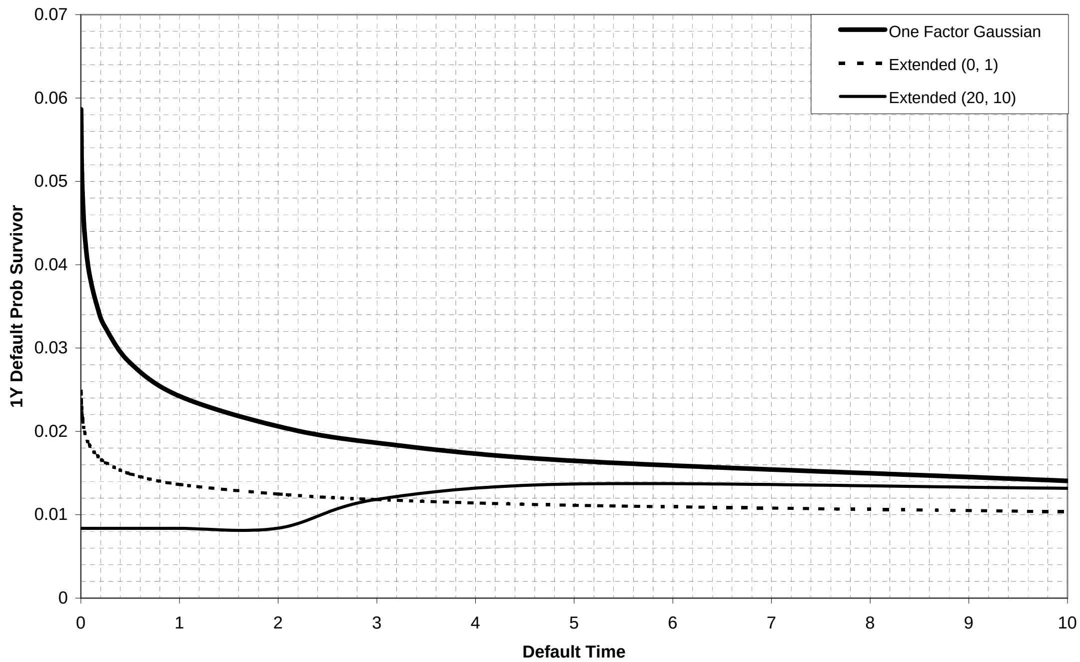

Figure 3: Conditional Probability of name 1 defaulting before time t+1Y, given that it has survived up to time t and a name 2 has defaulted before time t.

From the figure one can observe totally different correlation effects. The standard copula model behaves as expected where a fast default event causes a large shock in the default probability of the surviving name. In the limit, when $t \downarrow 0$ the conditional default probability of the surviving name will be tend to 1, i.e. certain default. When the default event occurs further in the future, the effect of the default event reduces.

For the extended model the effect largely depends on the distribution of the external default risk factor, Z. For the case of a standard normal distribution for Z, the results look similar to the standard factor Gaussian copula case, but with a far lower shock. Here one should note that the distributions of both the market risk factor, X and that of the external default risk factor, Z are equal. Thus an instantaneous default event is equally likely to be caused by either the market risk factor or by the external risk factor.

In case Z has a normal distribution with mean 20 and standard deviation 10, the behavior is totally different. This is caused by the fat tails of this distribution, compared to the standard normal distribution. One can observe that the default probability of the surviving name is hardly affected by a default event of the other name within 1 year. After this period the effect slowly increases. This can be explained by the fact that, under these settings, a fast default event can be attributed to an external cause. Intuitively this latter behavior is preferred. Both names have an initial CDS spread of only $50BP$ and thus are not expected to default. If, however, a default event would occur within a year, it is not likely that the state of the economy has dramatically worsened. An external default event is much more likely and this will not affect the default probability of the surviving name.

When taking a closer look at the model in (3) one can conclude that in case σ is strictly larger than 1 an instantaneous default, i.e. a default before time $t \downarrow 0$, is caused by an external shock and thus has no influence whatsoever on the default probability of the surviving names. Thus with $\sigma > 1$, all graphs as in figure 3 will start at $83BP$. The behavior for larger values for t depends on the correlation and the parameters for the external risk factor, μ and σ.

5.2 Model-Implied Base Correlations

Next we consider the Base Correlation skew which is generated by the extended model. An example of the Base Correlation skew observed in the market was given in figure 2. In this figure a close to linear relationship between detachment level and Base Correlation could be observed. Any practical model should thus be able to generate a curve.

In section 4 it was argued that the inclusion of an external default risk factor provides a plausible explanation for an upward sloping Base Correlation skew. However it is not clear whether this curve will be close to linear. In order to investigate this, some tests are performed in this section. The model in (3) is used as input to generate quotes for tranche default swaps with detachments at $3\%, 7\%, 10\%, 15\%$ and 30%. These model quotes are then used as input to the Base Correlation method and the resulting Base Correlation skew is analyzed.

As discussed in section 4 the Base Correlations generated by the extended model are bounded from above by the correlation used as input, ρ^2. Thus judging from figure 2 a correlation of at least 70% is required.

The following figure shows the Base Correlation skews implied by our model for different settings. First we consider the case of a standard normal distribution for the external default risk factor. Thus a default event is equally likely to be caused by the market risk factor as by the external default risk factor. In this case the correlation has been set to 70%. In addition, another case is considered where correlation has been set to 90%, $\mu = 1.9$ and $\sigma = 2$. These parameters have been chosen such that the implied Base Correlation skew closely matches the skew of the market at both endpoints.

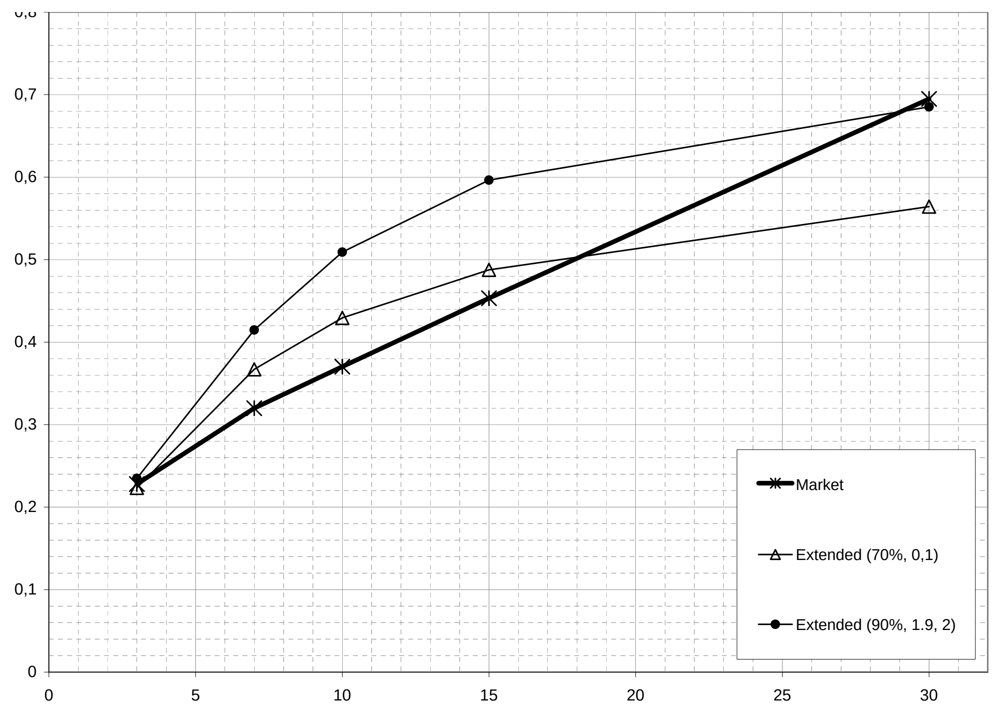

Figure 4: The model-implied Base Correlation skew for two different settings. In addition the skew as observed in the market is given.

From the figure one can see that the extended model is able to generate an upward sloping skew. However the relationship does not seem to be close to linear. When investigating other parameter combinations it appears that not much variation in the shape of the curve can be generated. In all cases we obtain a slightly concave curve. When increasing volatility of the external default risk factor, σ, one brings the curve down. The same holds when decreasing the mean of the external risk factor, μ. At the same time the curve becomes steeper. In extreme cases where the external risk factor is dominant, the curve will be close to zero. In case the market risk factor is dominant, the curve will be flat and close to ρ^2.

6 Further Applications

This model has straightforward applications for the pricing of liquid basket credit derivatives such as tranches default swaps. As was shown in the previous section, the model results in much more realistic correlation effects for instantaneous defaults and at the same time is able to explain the upward sloping Base Correlations observed in the market. Due to these nice properties of the model, it is likely that it will provide a good starting point for modelling more complex products with baskets as underlyings and option-like payoffs, such as credit default swaptions on certain tranches. These next generation products are likely to have a short time to maturity, making the standard model as well as simple extensions useless, due to the problem of fast default events.

Furthermore the model can be used to determine prices for a CDO of CDOs, or CDO^2. Due to the large complexity of this product one does not have the semi-analytical tractability. Instead one should rely on Monte Carlo Simulations and apply a look-through approach. However the Base Correlation method is not modelling the default behavior, but rather the loss distribution of a basket. It is not clear how an observed skew should be translated to a proper model for the correlation structure, which is required for valuing these products.

7 Conclusion

Over the last years trading in tranche default swaps has undergone a rapid development and with it the research efforts for valuing these products. A widely used model for imposing default correlation is the one-factor Gaussian copula approach. In this paper we have addressed two mayor weaknesses of the standard one factor Gaussian copula. First it was shown that the standard model has highly undesirable correlation effects in case of fast default events. In case a name with small default probability defaults instantaneously and thus unexpectedly, the default distributions of the remaining names are affected dramatically. However a sudden unexpected default event is likely to have external causes, rather than being caused by an extremely bad state of the entire economy. The standard factor Gaussian copula is not able to model this realistically.

In addition it is well known that the model is incapable to explain observed market quotes for CDO tranches. Due to increased liquidity of CDO tranches, it is of crucial importance to use a model which is able to explain the observed quotes.

It was concluded that linearly combining the common factor and idiosyncratic terms was the cause of extreme jumps of survivors at a fast default event of a correlated name. We have proposed a simple model which overcomes this problem, but maintains a large degree of analytical tractability. By focussing on the minimum of two terms, one incorporating market risk while the other can be regarded as a measure for external risk, this undesired behavior can be avoided. The model allows for default events being totally external and thus do not reveal any additional information about the common factor. By carefully choosing model parameters, one can determine the probability that a default event over a certain amount of time has an external cause. With this setup one can thus tackle the problem of absurd correlation effects caused by instantaneous default events. This model can consequently be used as a starting point for modelling more complex credit derivatives, involving option-like payoffs.

In addition it was shown that the lack of totaly external default risk in the standard model provides a plausible explanation for the Base Correlation skew observed in the market. Due to the inclusion of totally external causes for default events, the model allows for more independence for low subordination levels. As the number of defaults increase, it is more likely that the realization of the common factor has been low, thus having an increasing effect on correlation. Combining these observations explains increasing correlations for increasing detachment levels.

Moreover one can link the extra parameters directly to the shape of the base correlation skew. When given a Base Correlation skew, one should use the largest default correlation of the skew as input to the model. By choosing appropriate parameters for the distribution of the external default risk factor, one can determine the slope of the Base Correlation skew.

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